Curvature

Introduction

Curvature is a geometrical property of a curve. It is basically a measure of how curved a curve is. Hence the curvature depends only on the shape of the concerned curve. For example, large circles have smaller curvature as compared to the small circles which bend more sharply. Again a straight line has zero curvature. In this chapter various formulae of curvature are established based on different analytical forms of curves. The radius of curvature, centre of curvature, circle of curvature, evolute of a curve are also discussed.

8.1 Simple Definition

The curvature of a curve at a point is the rate of change of direction of the curve at that point with respect to the arc. In other words the curvature of a curve is the "rate at which the curve curves".

8.2 Angle of Contingence and Geometric Definition of Curvature

With reference to the co-ordinate axes (Ox, Oy) let an arc of a given curve $y = f(x)$ be measured from a fixed point A on the curve. Let P and Q be two neighbouring points on the curve $y = f(x)$. Draw two tangents KPL and MQN at P and Q respectively which intersect at T and meet x-axis at K and M. Let arc AP $= s$ and arc AQ $= s + \Delta s$, so that arc PQ $= \Delta s$ (see **Fig. 8.1**). Let tangents at P and Q make angles ψ and $\psi + \Delta\psi$ with the positive direction of x-axis. Then the angle $\angle$NTL between the tangents is called **the angle of contingence** or **the angle of incidence** of the arc PQ. So the angle of contingence of any arc is just the difference of the angles which the tangents at its extremities make with any given fixed straight line.

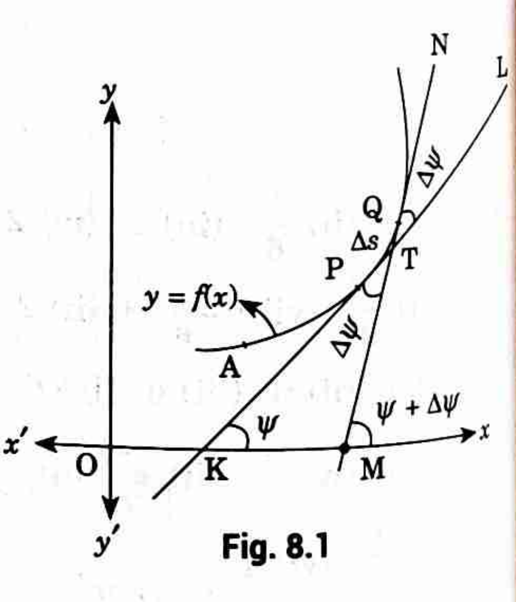


Fig. 8.1

The ratio $\frac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ at the point P. As the point $Q \to P$ along the curve, the limiting value : $\lim_{Q \to P} \frac{\Delta\psi}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s}$ [since $\Delta s \to 0$, as $Q \to P$]

$$= \frac{d\psi}{ds} \text{ (say)}$$

is called the **curvature** of the curve at the point P.

Note

The reciprocal of curvature at P *i.e.*, $\frac{ds}{d\psi}$ is called the radius of curvature of the concerned curve at P and denoted by ρ.

8.3 Geometry behind Radius of Curvature

Let the two normals at points P and Q close to each other meet at C′ on the normal at P. Now we make $Q \to P$ along the curve, consequently the intersecting point of normals C′ approaches to a definite point C(say) on the normal at P (see **Fig. 8.2**). Thus, in the limiting sense C is the ultimate point of intersection of two normals indefinitely close together. The length PC is the radius of curvature of the given curve at the point P.

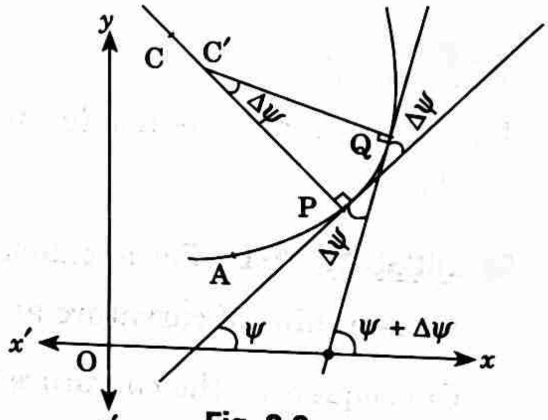


Fig. 8.2

8.4 Radius of Curvature of a Circle

Let APQ be a circle with centre at C and radius a. P, Q are two neighbouring points on the circle. Angle between tangents at P and Q is $\Delta\psi$ so that angle between normals at P and Q is also $\Delta\psi$ at the centre C(see **Fig. 8.3**).

Now let arc AP = s, arc AQ = $s + \Delta s$, so that arc PQ = $\Delta s = a\Delta\psi$. Therefore, the curvature of the circle at the point P

$$= \lim_{\Delta\psi \to 0}\left(\frac{\Delta\psi}{\Delta s}\right) = \lim_{\Delta\psi \to 0}\left(\frac{\Delta\psi}{a\Delta\psi}\right) = \frac{1}{a} \quad \text{[since } \Delta s = a\Delta\psi\text{]}$$

$\Rightarrow \frac{d\psi}{ds}$, the curvature = $\frac{1}{a}$

$\Rightarrow$ the radius of curvature of the circle with radius 'a' at any point P on the circle is $\frac{ds}{d\psi}$ = 'a'.

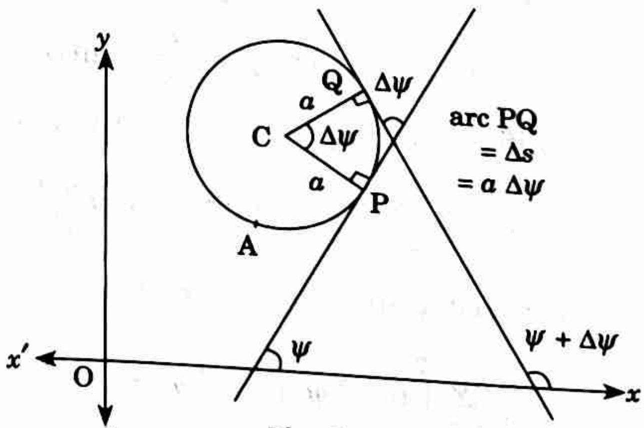


Fig. 8.3

8.5 Various Formulae connected to Radius of Curvature

A. For the Intrinsic Equation : $s = f(\psi)$

We have already derived the formula $\rho = \frac{ds}{d\psi} = \lim_{\Delta\psi \to 0} \frac{\Delta s}{\Delta\psi}$ ...(1)

where ρ is the radius of curvature at any point of the concerned curve and Δs, the small arc length between two neighbouring points on the curve and $\Delta\psi$, the small angle between two tangents drawn at the said two neighbouring points.

Note

The above formula is helpful, if the equation of the curve is given in the intrinsic form : $s = f(\psi)$.

■ **Illustration-1 :** For a catenary $s = c\tan\psi$, we have by **Formula** (1) $\rho = \frac{ds}{d\psi} = c\sec^2\psi$ is the radius of curvature at any point on the curve.

Consequently, the curvature $= \frac{1}{\rho} = \frac{1}{c\sec^2\psi} = \frac{1}{c(1+\tan^2\psi)} = \frac{1}{c\left(1+\frac{s^2}{c^2}\right)} = \frac{c}{s^2+c^2}$,

which is the rate of its deflection at any point on the curve.

B. For the equations : $x = x(s)$ and $y = y(s)$

We have the following relations :

$$\sin\psi = \frac{dy}{ds},\ \cos\psi = \frac{dx}{ds},\ \tan\psi = \frac{dy}{dx}$$

Differentiating first two relations with respect to s, we have

$$\cos\psi\frac{d\psi}{ds} = \frac{d^2y}{ds^2} \text{ and } -\sin\psi\frac{d\psi}{ds} = \frac{d^2x}{ds^2}.$$

Squaring and adding we get

$$\left(\frac{d\psi}{ds}\right)^2(\cos^2\psi + \sin^2\psi) = \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2$$

$$\Rightarrow \boxed{\frac{1}{\rho^2} = \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2x}{ds^2}\right)^2} \quad \left[\text{as } \rho = \frac{ds}{d\psi}\right] \quad ...(2)$$

Again, $\dfrac{-\sin\psi\dfrac{d\psi}{ds}}{\sin\psi} = \dfrac{\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}} \Rightarrow \boxed{\dfrac{d\psi}{ds} = \dfrac{1}{\rho} = \dfrac{-\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}}}$ ... (3

Similarly, $\boxed{\dfrac{1}{\rho} = \dfrac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}}}$ ... (4

Note

Formulae (2) – (4) are suitable when the equation of a curve is given as $x = x(s)$ and $y = y(s)$.

C. For the Cartesian Equation : $y = f(x)$

Differentiating the relation $\tan\psi = \dfrac{dy}{dx}$

with respect to x we get $\sec^2\psi\dfrac{d\psi}{dx} = \dfrac{d^2y}{dx^2}$ ... (4a)

Again, $\dfrac{d\psi}{dx} = \dfrac{d\psi}{ds}\cdot\dfrac{ds}{dx} = \dfrac{1}{\rho}\cdot\dfrac{1}{\cos\psi}$

Thus $\sec^3\psi\cdot\dfrac{1}{\rho} = \dfrac{d^2y}{dx^2}$ [using (4a)]

$$\Rightarrow \rho = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\psi\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \left[\text{since } \tan\psi = \frac{dy}{dx}\right]$$

$$\Rightarrow \boxed{\rho = \left|\frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}\right| \quad \left[y_1 = \frac{dy}{dx},\, y_2 = \frac{d^2y}{dx^2} \neq 0\right]} \quad \text{... (5)}$$

D. For the Cartesian Equation : $x = g(y)$

If the tangent at a point to the curve becomes parallel to y-axis, y_1 *i.e.*, $\frac{dy}{dx}$ becomes infinity, so formula (5) fails. Further if equation of the curve be given in the form $x = g(y)$, we will use following formula [by interchanging x and y in formula (5)]. Thus,

$$\rho = \left|\frac{\left(1+x_1^2\right)^{\frac{3}{2}}}{x_2}\right|, x_1 = \frac{dx}{dy}, x_2 = \frac{d^2x}{dy^2} \neq 0. \quad \ldots (6)$$

Note

Formula (6) is useful, when x is given explicitly in terms of y.

E. For the Implicit Equation : $f(x, y) = 0$

If the equation of the curve be given as $f(x, y) = 0$, *i.e.*, as **implicit function** and $\frac{\partial f}{\partial y}(x, y) = f_y(x, y) \neq 0$, we have

$\frac{dy}{dx} = y_1 = -\frac{f_x}{f_y}$. We differentiate it w.r.t x partially and obtain

$f_{xx} + 2f_{xy} \cdot y_1 + f_{yy}\, y_1^2 + f_y\, y_2 = 0$ [assuming $f_{xy} = f_{yx}$].

Thus substituting the values of y_1 and y_2 in formula (5) we get

$$\rho = \left|\frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}\right| = \left|-\frac{\left(1+\frac{f_x^2}{f_y^2}\right)^{\frac{3}{2}} \cdot f_y}{f_{xx} + 2f_{xy}\left(-\frac{f_x}{f_y}\right) + f_{yy}\left(-\frac{f_x}{f_y}\right)^2}\right| \Rightarrow \boxed{\rho = \left|\frac{\left(f_x^2+f_y^2\right)^{\frac{3}{2}}}{f_{xx}f_y^2 - 2f_{xy}f_x f_y + f_{yy}f_x^2}\right|} \quad \ldots (7)$$

(assuming the denominator in the formula (7) does not vanish at the concerned point)

F. For the Parametric Equations : $x = f(t), y = g(t)$

If the equation of the curve be given as **parametric equations** : $x = f(t)$, $y = g(t)$, where t be the parameter.

Then $x' = f'(t) = \dfrac{dx}{dt}$ and $y' = g'(t) = \dfrac{dy}{dt}$,

so that $y_1 = \dfrac{dy}{dx} = \dfrac{y'}{x'}$ and $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{dt}{dx}$

$$= \frac{x'y'' - y'x''}{x'^2}\cdot\frac{1}{x'} \quad \left[y'' = \frac{d^2y}{dt^2} \text{ and } x'' = \frac{d^2x}{dt^2}\right]$$

Therefore, on using formula (5), we get $\boxed{\rho = \left|\dfrac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''}\right|}$ ... (8)

[assuming the denominator in the formula (8) does not vanish at the concerned point and suffixes denote order of differentiation with respect to "t"].

■ **Illustration-4** : The parametric equation of the parabola $y^2 = 4ax$ is $x = at^2, y = 2at$. The radius of curvature of the parabola at $(a, 2a)$ *i.e.*, at $t = 1$ can be evaluated as follows :

$$x' = \frac{dx}{dt} = 2at,\ x'' = \frac{d^2x}{dt^2} = 2a \text{ and } y' = \frac{dy}{dt} = 2a,\ y'' = \frac{d^2y}{dt^2} = 0$$

Thus, $\rho(a, 2a) = \rho\big|_{t=1} = \left|\dfrac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''}\right|_{t=1} = \left|\dfrac{\left\{(2at)^2 + (2a)^2\right\}^{\frac{3}{2}}}{-4a^2}\right|_{t=1} = 2a\left(t^2+1\right)^{\frac{3}{2}}\Big|_{t=1} = 2^{\frac{5}{2}}a$.

G. For the Polar Equation : $r = f(\theta)$

If the equation of the curve is given as **Ploar equation $r = f(\theta)$**, then the radius of curvature ρ takes the following form

$$\boxed{\rho = \left|\frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}\right|},\ \left[r_1 = \frac{dr}{d\theta} \text{ and } r_2 = \frac{d^2r}{d\theta^2}\right] \quad \text{... (9)}$$

(assuming the denominator in (9) does not vanish at the concerned point)

■ **Illustration-5** : Suppose the polar equation of a curve is $r = a\cos\theta$. The radius of curvature at $\left(0, \frac{\pi}{2}\right)$ can be found out as follows :

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta,\ r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

Thus, at $\left(0, \frac{\pi}{2}\right)$ $r = 0$, $r_1 = -a$ and $r_2 = 0$. So the radius of curvature at $\left(0, \frac{\pi}{2}\right)$ is

$$\rho = \left|\frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}\right| = \frac{a^3}{2a^2} = \frac{a}{2}.$$

H. For the Polar Equation : $u = f(\theta)$, $u = \frac{1}{r}$

If the equation of the curve be given as $u = f(\theta)$, $u = \frac{1}{r}$, we have

$$\rho = \left|\frac{\left(u^2 + u_1^2\right)^{\frac{3}{2}}}{u^3(u + u_2)}\right|,\ u_1 = \frac{du}{d\theta},\ u_2 = \frac{d^2u}{d\theta^2}. \quad \text{... (10)}$$

[assuming the denominator in (10) does not vanish].

I. For the Pedal Equation : $p = f(r)$

Here $\psi = \theta + \phi$,

$\sin\phi = r\frac{d\theta}{ds}$, $\cos\phi = \frac{dr}{ds}$,

$\tan\phi = r\frac{d\theta}{dr}$ and $p = r\sin\phi$ [see **Fig. 8.4**]

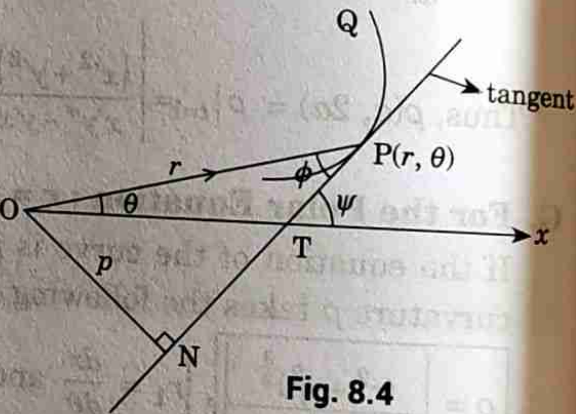


Fig. 8.4

$$\Rightarrow \frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{d\phi}{dr}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\frac{d\psi}{ds} = r\cdot\frac{1}{\rho} \left[\text{since } \theta + \phi = \psi \text{ and } \frac{d\psi}{ds} = \frac{1}{\rho}\right]$$

$$\Rightarrow \boxed{\rho = r\frac{dr}{dp}} \quad \text{... (11)}$$

■ **Illustration-6** : The radius of curvature ρ of the curve whose pedal equation is $r^4 = a^3p$ be given by

$$\rho = r\frac{dr}{dp} = r \times \frac{a^3}{4r^3} = \frac{a^3}{4r^2} \left[\text{since } 4r^3\frac{dr}{dp} = a^3\right]$$

J. For the Tangential Polar Equation : $p = f(\psi)$

Clearly, $\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi} = \frac{dp}{dr}\cdot\cos\phi\cdot\rho$ $\left[\text{since } \frac{ds}{d\psi} = \rho\right]$

$$= \frac{dp}{dr}\cdot\cos\phi\cdot\left(r\frac{dr}{dp}\right) \left[\text{since } \rho = r\frac{dr}{dp}\right]$$

$$= r\cos\phi$$

Again, $p = r\sin\phi$.

Thus, $p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2$

$$\Rightarrow 2p + 2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp} = 2\rho \text{ [by differentiating with respect to } p]$$

$$\Rightarrow \boxed{\rho = p + \frac{d^2p}{d\psi^2}} \quad \dots (12)$$

Illustration-7 : For the curve $p = a(1 + \sin\psi)$ [tangential polar equation], we have

$$\frac{dp}{d\psi} = a\cos\psi \Rightarrow \frac{d^2p}{d\psi^2} = -a\sin\psi = a - p \text{ [since } p = a + a\sin\psi]$$

$$\Rightarrow p + \frac{d^2p}{d\psi^2} = a \Rightarrow \rho = a$$

Hence the radius of curvature is a.

(see **Fig. 8.6**). Accordingly, the curvatures of the curve and the circle at the point of contact are same. It is therefore acceptable to describe the curvature of a curve at a given point with reference to a circle thus drawn. Hence we consider the existence of such a circle, for each point of a curve, called **circle of curvature** of that point. The centre and radius of this circle are called **centre of curvature** and **radius** respectively. Any chord of this circle drawn through the point of contact in any direction is said to be the **chord of curvature** in that direction.

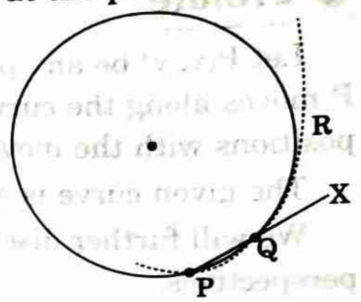


Fig. 8.6

Note

The equation of the circle of curvature corresponding to any point P(x, y) on the curve is $(x-\bar{x})^2+(y-\bar{y})^2=\rho^2$ with $(\bar{x},\bar{y})$ as centre of curvature and ρ as radius of curvature.

8.8 Centre of Curvature

Let $C(\bar{x}, \bar{y})$ be the centre of curvature corresponding any point $P(x, y)$ on the curve $f(x, y) = 0$, we have the equation of the normal at $P(x, y)$ is $(Y - y)y_1 + (X - x) = 0$ [where $y_1 = \dfrac{dy}{dx}$]

Since centre of curvature $C(\bar{x}, \bar{y})$ lies on the normal, so we get

$$(\bar{y} - y)y_1 + (\bar{x} - x) = 0 \Rightarrow \bar{x} - x = -(\bar{y} - y)y_1 \quad \text{... (18)}$$

Hence PC = ρ[radius of curvature at $P(x, y)$]

$$\Rightarrow PC^2 = \rho^2 \Rightarrow (\bar{x} - x)^2 + (\bar{y} - y)^2 = \rho^2 = \frac{(1+y_1^2)^3}{y_2^2}$$

$$\Rightarrow (\bar{y} - y)^2 . y_1^2 + (\bar{y} - y)^2 = \frac{(1+y_1^2)^3}{y_2^2} \quad \text{[using (18)]}$$

$$\Rightarrow (\bar{y} - y)^2(1 + y_1^2) = \frac{\left(1+y_1^2\right)^3}{y_2^2}$$

$$\Rightarrow \bar{y} - y = \frac{1+y_1^2}{y_2} \Rightarrow \bar{y} = y + \frac{1+y_1^2}{y_2} \quad \text{... (19)}$$

Now from (18), $\bar{x} - x = -\dfrac{(1+y_1^2)}{y_2}y_1$

$$\Rightarrow \bar{x} = x - \frac{(1+y_1^2)y_1}{y_2} \quad \text{... (20)}$$

From (19) and (20) we get the formulae for centre of curvature as follows :

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}, \ \bar{y} = y + \frac{1+y_1^2}{y_2}$$

Evolute

Let P(x, y) be any point on a given curve and C $(\bar{x}, \bar{y})$ be the centre of curvature at P. As P moves along the curve, the point C also changes its position. Thus, the locus of C (several positions with the movement of P) is called the **evolute** of the given curve.

The given curve is sometimes called an **involute** to the evolute so obtained.

We will further discuss **Evolute** and **Involute** in Envelopes (**Chapter 10**) with different perspectives.

1. **Find the radius of curvature at the point (s, ψ) on the following curves :**

(i) $s = 8a \sin^2 \frac{\psi}{6}$, (ii) $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$.

Solution (i) Here equation of the curve is $s = 8a \sin^2 \frac{\psi}{6} = 4a\left(1 - \cos\frac{\psi}{3}\right)$

Now Differentiating with respect to ψ, we have

$$\frac{ds}{d\psi} = 4a\left(0 + \frac{1}{3}\sin\frac{\psi}{3}\right) = \frac{4a}{3}\sin\frac{\psi}{3}$$

Therefore, the radius of curvature is $\rho = \frac{ds}{d\psi} = \frac{4a}{3}\sin\frac{\psi}{3}$.

(ii) Here equation of the curve is $s = a \log \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$.

Now differentiating with respect to ψ, we get

$$\frac{ds}{d\psi} = a.\frac{1}{\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}.\sec^2\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cdot\frac{1}{2} = a.\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)} = \frac{a}{\sin\left(\frac{\pi}{2}+\psi\right)} = a\sec\psi$$

Therefore, the radius of curvature is $\rho = \frac{ds}{d\psi} = a\sec\psi$.

4. **Find the radius of curvature at the point (3, 4) on the curve $xy = 12$.**

[CU 2008]

Solution Here the equation of the curve is $xy = 12$ or $y = \frac{12}{x}$ ... (1)

Now differentiating (1) with respect to x, we get $\frac{dy}{dx} = y_1 = -\frac{12}{x^2}$.

Again differentiating with respect to x, we get $\frac{d^2y}{dx^2} = y_2 = \frac{24}{x^3}$.

We know that the radius of curvature $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$,

Now $(y_1)_{(3,\,4)} = -\frac{12}{3\times 3} = -\frac{4}{3}$ and $(y_2)_{(3,\,4)} = \frac{24}{3^3} = \frac{8}{9}$.

Therefore, ρ at $(3, 4) = \frac{\left[1+\left(-\frac{4}{3}\right)^2\right]^{\frac{3}{2}}}{\frac{8}{9}} = \frac{9}{8}\left(1+\frac{16}{9}\right)^{\frac{3}{2}} = \frac{9}{8}\left(\frac{25}{9}\right)^{\frac{3}{2}} = \frac{9}{8}\left(\frac{5}{3}\right)^{2\times\frac{3}{2}} = \frac{9}{8}\times\frac{5^3}{3^3} = \frac{5^3}{8\times 3} = \frac{125}{24}$.

7. **Find the radius of curvature of the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$; at $\theta = \frac{\pi}{4}$.**

Solution Here equation of the curve is $\left.\begin{array}{l} x = a\cos^3\theta \\ y = a\sin^3\theta \end{array}\right\}$ ... (1)

Now $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Therefore, $y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = -\tan\theta.$

Now $y_2 = \frac{d^2y}{dx^2} = -\sec^2\theta \cdot \frac{d\theta}{dx} = -\frac{\sec^2\theta}{\frac{dx}{d\theta}} = -\frac{\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\sin\theta\cos^4\theta}.$

Thus at $\theta = \frac{\pi}{4}$, $y_1 = -\tan\frac{\pi}{4} = -1$ and $y_2 = \frac{1}{3a\sin\frac{\pi}{4}\cos^4\frac{\pi}{4}} = \frac{1}{3a\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{4}} = \frac{4\sqrt{2}}{3a}.$

We know that radius of curvature $= \rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}.$

Hence at $\theta = \frac{\pi}{4}$, $\rho = \frac{(1+1)^{\frac{3}{2}}}{\frac{4\sqrt{2}}{3a}} = \frac{3a\cdot 2\sqrt{2}}{4\sqrt{2}} = \frac{3a}{2}.$

9. **Find the radius of curvature at any point (r, θ) for the curve $r = a(1 - \cos\theta)$.**

[NBH 2002, KH 2011]

Solution Here the equation of the curve is $r = a(1 - \cos\theta)$ ... (1)

Differentiating (1) with respect to θ, we get $\frac{dr}{d\theta} = r_1 = a\sin\theta$.

Similarly, $\frac{d^2r}{d\theta^2} = r_2 = a\cos\theta$.

$$\text{The radius of curvature } \rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} = \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)a\cos\theta}$$

$$= \frac{a^3(2-2\cos\theta)^{\frac{3}{2}}}{a^2[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta]}$$

$$= \frac{a(2-2\cos\theta)^{\frac{3}{2}}}{(3-3\cos\theta)} = \frac{a\cdot 2\sqrt{2}}{3}\sqrt{1-\cos\theta} = \frac{2\sqrt{2}a}{3}\cdot\sqrt{\frac{r}{a}} = \frac{2}{3}\sqrt{2ar}\,.$$

11. **Show that in any curve**

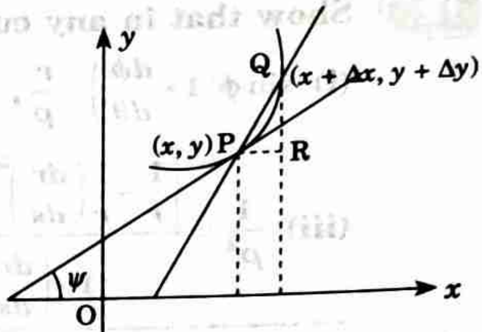
(i) $\rho^2 = \left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2$, **(ii)** $\frac{1}{\rho} = \frac{d^2x}{ds^2} \Big/ \frac{dy}{ds} = \frac{d^2y}{ds^2} \Big/ \frac{dx}{ds}$, **(iii)** $\frac{1}{\rho^3} = \frac{d^2x}{ds^2} \cdot \frac{d^3y}{ds^3} - \frac{d^2y}{ds^2} \cdot \frac{d^3x}{ds^3}$.

Solution We know for any curve $y = f(x)$,

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2},\ \frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2},$$ so that $(ds)^2 = (dx)^2 + (dy)^2$ (see the adjoining figure).

Also $\cos\psi = \frac{dx}{ds}$, $\sin\psi = \frac{dy}{ds}$, $\tan\psi = \frac{dy}{dx}$ and $\rho = \frac{ds}{d\psi}$, where symbols have their usual meaning.

(i) RHS $= \left(\dfrac{dx}{d\psi}\right)^2 + \left(\dfrac{dy}{d\psi}\right)^2 = \left(\dfrac{dx}{ds}\cdot\dfrac{ds}{d\psi}\right)^2 + \left(\dfrac{dy}{ds}\cdot\dfrac{ds}{d\psi}\right)^2$



$$= \left(\frac{ds}{d\psi}\right)^2\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right] = \rho^2(\cos^2\psi + \sin^2\psi)$$

$$= \rho^2 \left[\text{since } \rho = \frac{ds}{d\psi},\ \cos\psi = \frac{dx}{ds},\ \sin\psi = \frac{dy}{ds}\right]$$

= LHS **[Proved]**

(ii) We know $\dfrac{dx}{ds} = \cos\psi$

$$\Rightarrow \frac{d^2x}{ds^2} = -\sin\psi\cdot\frac{d\psi}{ds} \quad \text{[by differentiating with respect to } s] \quad \ldots (1)$$

$$= -\frac{dy}{ds} / \frac{ds}{d\psi} \quad \left[\text{since } \sin\psi = \frac{dy}{ds}\right]$$

$$= -\frac{dy}{ds}\cdot\frac{1}{\rho} \quad \left[\text{since } \rho = \frac{ds}{d\psi}\right]$$

$$\Rightarrow \frac{1}{\rho} = \frac{d^2x}{ds^2} / \frac{dy}{ds}$$ [in magnitude] **[Proved]**

Similarly, differentiating the relation $\dfrac{dy}{ds} = \sin\psi$ with respect to s, we have

$$\frac{d^2y}{ds^2} = \cos\psi\cdot\frac{d\psi}{ds} \quad \ldots (2)$$

$$= \frac{dx}{ds} / \frac{ds}{d\psi} = \frac{dx}{ds}\cdot\frac{1}{\rho}$$

$$\Rightarrow \frac{1}{\rho} = \frac{d^2y}{ds^2} / \frac{dx}{ds}.$$ **[Proved]**

(iii) Dividing (1) by (2) we have, $\dfrac{d^2x}{ds^2} / \dfrac{d^2y}{ds^2} = -\tan\psi$.

Differentiating both sides with respect to s, we obtain

$$\frac{\dfrac{d^2y}{ds^2}\cdot\dfrac{d^3x}{ds^3} - \dfrac{d^2x}{ds^2}\cdot\dfrac{d^3y}{ds^3}}{\left(\dfrac{d^2y}{ds^2}\right)^2} = -\sec^2\psi\cdot\frac{d\psi}{ds}$$

$$\Rightarrow \frac{d^2x}{ds^2}\cdot\frac{d^3y}{ds^3} - \frac{d^2y}{ds^2}\cdot\frac{d^3x}{ds^3} = \sec^2\psi\cdot\frac{d\psi}{ds}\cdot\left(\cos\psi\frac{d\psi}{ds}\right)^2 \quad \text{[by (2)]}$$

$$= \left(\frac{d\psi}{ds}\right)^3 = \frac{1}{\rho^3}$$

$$\Rightarrow \frac{1}{\rho^3} = \frac{d^2x}{ds^2}\cdot\frac{d^3y}{ds^3} - \frac{d^2y}{ds^2}\cdot\frac{d^3x}{ds^3}.$$ **[Proved]**

19. **Find the radius of curvature of the curve $y = xe^{-x}$ at the point where y is maximum.** **[CH 2014, 2019; BH 2005, 2009, 2014, 2011; KH 2006]**

Solution We first find the point on the curve $y = xe^{-x}$ ... (1)

where y is maximum.

Differentiating (1) with respect to x, we get $y_1 = -xe^{-x} + e^{-x}$

Now $y_1 = 0$ gives $e^{-x}(1 - x) = 0 \Rightarrow x = 1$ [since $e^{-x} \neq 0$]

Now $y_2 = -e^{-x}(1 - x) - e^{-x} = -e^{-x}(2 - x)$

At $x = 1$, $y_2 = -e^{-1}(2-1) = -\frac{1}{e} < 0$

Thus y is maximum at $x = 1$.

Now, we have to find the radius of curvature at $x = 1$.

At $x = 1$, $y_1 = 0$, $y_2 = -\frac{1}{e}$.

Now $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{|y_2|}$. Hence at $x = 1$, $\rho = \frac{\left(1+0^2\right)^{\frac{3}{2}}}{\left|-\frac{1}{e}\right|} = e$.

22. **Prove that in any curve $\frac{d\rho}{ds} = \frac{3y_1y_2^2 - y_3(1+y_1^2)}{y_2^2}$ and show that at every point of a circle $3y_1y_2^2 = y_3(1 + y_1^2)$.**

Solution We know that the radius of curvature ρ of a curve is given by

$$\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}, \quad \left[y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2} \neq 0\right]$$

Now differentiating with respect to x, we obtain

$$\frac{d\rho}{dx} = \frac{y_2 \cdot \frac{3}{2}\left(1+y_1^2\right)^{\frac{1}{2}} \cdot 2y_1y_2 - \left(1+y_1^2\right)^{\frac{3}{2}} \cdot y_3}{y_2^2} \Rightarrow \frac{d\rho}{ds} \cdot \frac{ds}{dx} = \left(1+y_1^2\right)^{\frac{1}{2}} \times \frac{3y_1y_2^2 - (1+y_1^2).y_3}{y_2^2}$$

$$\Rightarrow \frac{d\rho}{ds} \cdot \left(1+y_1^2\right)^{\frac{1}{2}} = \left(1+y_1^2\right)^{\frac{1}{2}} \times \frac{3y_1y_2^2 - (1+y_1^2)y_3}{y_2^2} \quad \left[\text{since } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right]$$

$$\Rightarrow \frac{d\rho}{ds} = \frac{3y_1y_2^2 - y_3(1+y_1^2)}{y_2^2}. \text{ [Proved]}$$

We further know that the radius of curvature ρ of a circle is equal to its radius (which is constant). Hence

$$\frac{d\rho}{ds} = 0, \text{ for a circle}$$

$$\Rightarrow \frac{3y_1y_2^2 - y_3(1+y_1^2)}{y_2^2} = 0 \Rightarrow 3y_1y_2^2 = y_3(1 + y_1^2). \text{ [Proved]}$$

25. **For the curve $a^2y = x^3$, show that the centre of curvature (α, β) is given by**

$$\alpha = \frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right); \beta = \frac{5}{2}\frac{x^3}{a^2} + \frac{a^2}{6x}.$$

Solution $a^2y = x^3$ (given equation of the curve)

$$\Rightarrow a^2y_1 = 3x^2 \left(y_1 = \frac{dy}{dx}\right)$$

$$\Rightarrow a^2y_2 = 6x \left(y_2 = \frac{d^2y}{dx^2}\right)$$

Thus, $y_1 = \dfrac{3x^2}{a^2}$, $y_2 = \dfrac{6x}{a^2}$. Hence the centre of curvature (α, β) is given by

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{3x^2}{a^2}\left(1 + \frac{9x^4}{a^4}\right) \times \frac{a^2}{6x} = x - \frac{x}{2}\left(1 + \frac{9x^4}{a^4}\right) = \frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right)$$

and $$\beta = y + \frac{1+y_1^2}{y_2} = y + \left(1 + \frac{9x^4}{a^4}\right) \times \frac{a^2}{6x} = \frac{x^3}{a^2} + \frac{3x^3}{2a^2} + \frac{a^2}{6x} = \frac{5}{2}\frac{x^3}{a^2} + \frac{a^2}{6x}$$

Hence proved.

38. **The tangents at two points P and Q on the cycloid**

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

are at right angles. Show that if ρ_1 and ρ_2 be the radii of curvature at these points, then $\rho_1^2 + \rho_2^2 = 16a^2$.

[VH 2019]

Solution The radius of curvature ρ at any point θ of the given cycloid can be obtained as

$$\rho = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''} = \frac{\{a^2(1-\cos\theta)^2+a^2\sin^2\theta\}^{\frac{3}{2}}}{a^2\cos\theta(1-\cos\theta)-a^2\sin^2\theta} = \frac{\{a^2+a^2(\cos^2\theta+\sin^2\theta)-2a^2\cos\theta\}^{\frac{3}{2}}}{a^2(\cos\theta-\cos^2\theta-\sin^2\theta)}$$

$$= \frac{(2a^2)^{\frac{3}{2}}(1-\cos\theta)^{\frac{3}{2}}}{-a^2(1-\cos\theta)} = \frac{\left(2a^2\times 2\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}}}{-a^2\times 2\sin^2\frac{\theta}{2}} = \frac{-4^{\frac{3}{2}}}{2}\cdot\frac{a^3}{a^2}\sin\frac{\theta}{2} = 4a\sin\frac{\theta}{2}\ \text{[in magnitude]}$$

Let P and Q respectively represent points θ_1 and θ_2 on the cycloid.

The gradient of tangent at θ

$$= \frac{y'}{x'} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{a\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a\cdot 2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

According to the problem, the product of gradient at points θ_1 and θ_2 is -1.

$$\Rightarrow \cot\frac{\theta_1}{2}\cdot\cot\frac{\theta_2}{2} = -1 \Rightarrow \cos\frac{\theta_1}{2}\cdot\cos\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\cdot\sin\frac{\theta_2}{2} = 0 \Rightarrow \cos\left(\frac{\theta_1}{2}-\frac{\theta_2}{2}\right) = 0 \Rightarrow \frac{\theta_1}{2} = \frac{\pi}{2}+\frac{\theta_2}{2}$$

Now the radii of curvature at $P(\theta_1)$ and $Q(\theta_2)$ are $\rho_1 = 4a\sin\frac{\theta_1}{2}$ and $\rho_2 = 4a\sin\frac{\theta_2}{2}$ respectively.

Thus, $\rho_1^2 + \rho_2^2 = 16a^2\sin^2\frac{\theta_1}{2} + 16a^2\sin^2\frac{\theta_2}{2} = 16a^2\cos^2\frac{\theta_2}{2} + 16a^2\sin^2\frac{\theta_2}{2}$ $\left[\text{since } \frac{\theta_1}{2} = \frac{\pi}{2} + \frac{\theta_2}{2}\right]$

$\Rightarrow \rho_1^2 + \rho_2^2 = 16a^2$ **[Proved]**

39. **If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioide $r = a(1 + \cos\theta)$, which passes through the pole, prove that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.** **[VH 2019, 2015; BH 1999, CH 1998]**

Solution Let P_1OP_2 be a chord of the cardioide $r = a(1 + \cos\theta)$ and O be the pole. Let ρ_1 and ρ_2 be the radii of curvature of the cardioide at P_1 and P_2 respectively.

Let the vectorial angle of P_1 be θ. Then the vectorial angle of P_2 will be $\pi + \theta$.

Now ρ_1 = radius of curvature at $P_1(\theta)$

$= \frac{2}{3}\sqrt{2ar}$ [same result for the cardioide $r = a(1 - \cos\theta)$, which was proved earlier]

$= \frac{2}{3}\sqrt{2a \cdot a(1+\cos\theta)} = \frac{4a}{3}\cos\frac{\theta}{2}$

$\Rightarrow \rho_2$ = radius of curvature at $P_2\ (\pi + \theta) = \frac{4a}{3}\cos\left(\frac{\pi+\theta}{2}\right) = \frac{4a}{3}\sin\frac{\theta}{2}$

Hence $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$. **[Proved]**